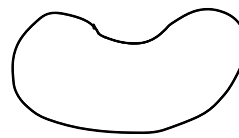
Let us compute the Jones polynomial for the unknot:

$$J(K_0; \lambda = 1) = d(1)^2 \, Z(K_0; 1)$$



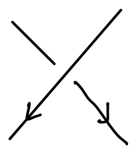
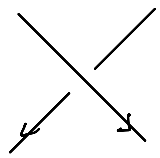
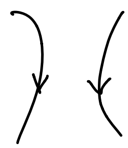
$K_0$

using $d(1) = Z^{-1}(K_0; 1)$

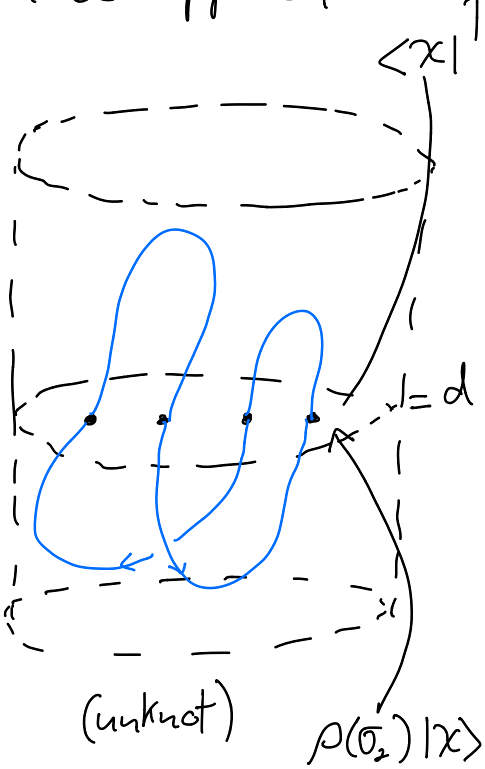
$$= Z^{-1}(K_0; 1)$$

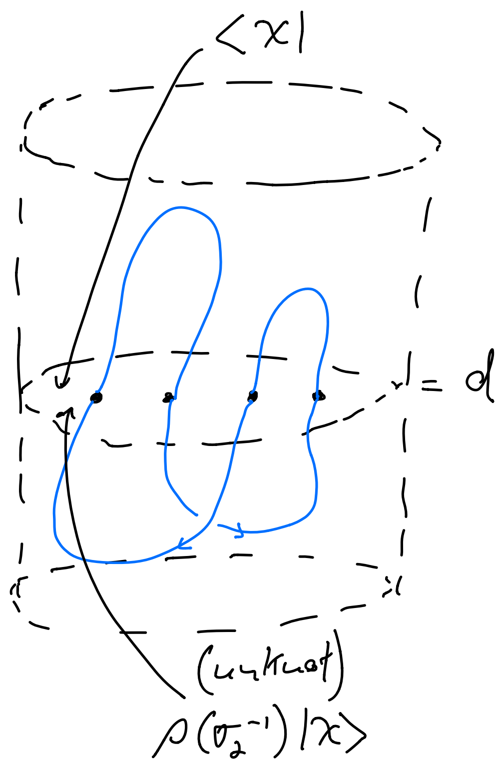
$$= d(1)$$

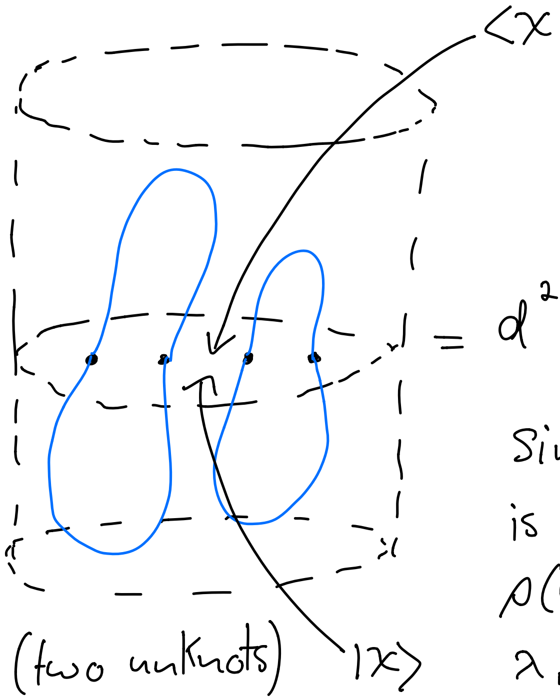
The skein relation (Prop. 2)

$q^{1/4}$  $- \; q^{-1/4}$  $= (q^{1/2} - q^{-1/2})$ 

can be applied as follows:






We have

$$\langle x|\rho(\sigma_2)|x\rangle = d$$

$$\langle x|\rho(\sigma_2^{-1})|x\rangle = d$$

$$\langle x|x\rangle = d^2$$

Since the Hilbert-space is two-dimensional, $\rho(\sigma_2)$ has two eigenvalues $\lambda_1$, $\lambda_2$

$$\longrightarrow \quad \rho(\sigma) - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2\,\rho(\sigma^{-1}) = 0$$

Taking the expectation value in $|x\rangle$ gives:

$$d - (\lambda_1 + \lambda_2)\,d^2 + \lambda_1\lambda_2\, d = 0$$

$$\longrightarrow \qquad d = \frac{1 + \lambda_1\lambda_2}{\lambda_1 + \lambda_2}$$

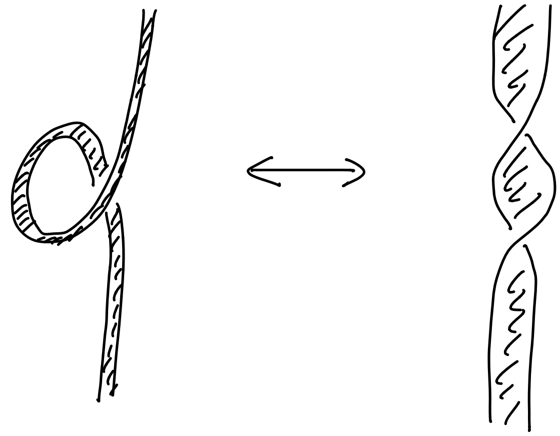
Using $\lambda_1 = -q^{-3/4}$, $\lambda_2 = q^{1/4}$, we get

$$d(1) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}} = 2\cos\left(\frac{\pi}{k+2}\right)$$

Alternatively, this could have been obtained from the skein relation for $P_L$:

$$q P_{L_+} - q^{-1} P_{L_-} = (q^{1/2} - q^{-1/2}) P_{L_0}\,, \quad P_0 = 1$$

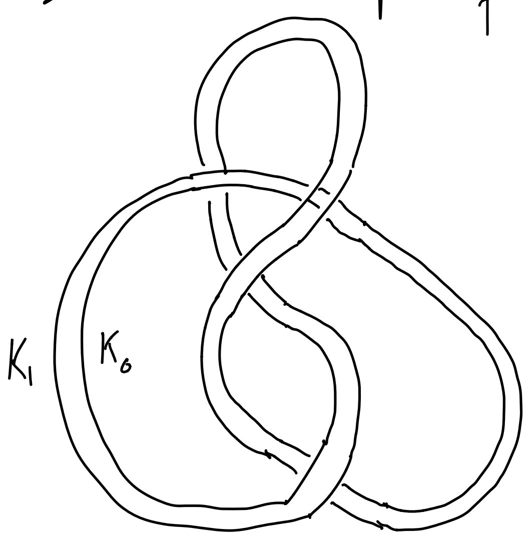
where $P_L = d(1)^{-1} \exp(-2\pi\sqrt{-1}\,\Delta, \omega(L))\, J_L$ and we use "blackboard framing" to compute $\omega(L)$:



$\longrightarrow$ increases framing by 1

Next, we want to compute $J(L; \lambda_1, \dots, \lambda_m)$ with several link components from $J_L$.

$\longrightarrow$ need concept of "cabling":




Let $K_0$ be an oriented framed knot with framing $t$. Take $K_1$ to be the companion knot on tubular bdr. of $K_0$ giving rise to framing $t$.

$\longrightarrow$ two-component link

We first compute $d(\lambda)$ for $\lambda > 1$:

<u>Lemma 2:</u>

$$d(\lambda) = \frac{q^{(\lambda+1)/2} - q^{-(\lambda+1)/2}}{q^{1/2} - q^{-1/2}}$$

where $q^{1/2} = \exp\left(\frac{2\pi\sqrt{-1}}{2(k+2)}\right)$

<u>Proof:</u>

We have $d(0) = 1$ and $d(1) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}}$ ✓

Let us now compute $d(\lambda)$ for $\lambda > 1$.

Consider the cabling for the trivial knot with 0 framing

$$\longrightarrow d(\lambda)\, d(\mu) = \sum_{\nu} N_{\lambda\mu}^{\nu}\, d(\nu) \qquad (*)$$

Observe that

$$\frac{q^{(\lambda+1)/2} - q^{-(\lambda+1)/2}}{q^{1/2} - q^{-1/2}} = \frac{S_{0\lambda}}{S_{00}}$$

where $S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin \frac{(\lambda+1)(\mu+1)}{k+2}$

$\longrightarrow$ It will be enough to show that $d(\lambda) = \frac{S_{0\lambda}}{S_{00}}$ satisfies $(*)$

This follows from the Verlinde formula (Prop. 6, §6):

$$N_{\lambda\mu\nu} = \dim \mathcal{H}(p_1, p_2, p_3 ; \lambda, \mu, \nu)$$
$$= \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}}$$

Namely,

$$\sum_{\nu} N^{\nu}_{\lambda\mu} d(\nu) = \sum_{\nu,\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \frac{S_{0\nu}}{S_{00}}$$
$$= \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} \delta_{0\alpha}}{S_{0\alpha} S_{00}} = \frac{S_{\lambda 0}}{S_{00}} \frac{S_{\mu 0}}{S_{00}}$$

$\square$

Let $K_0$ be an oriented framed knot and let $K_0 \cup K_1$ be a link obtained by cabling of $K_0$. We have:

<u>Lemma 3:</u>

The invariant $J(K_0, K_1 ; \lambda, \mu)$ of the link $K_0 \cup K_1$ obtained as a cabling of $K_0$ satisfies $J(K_0, K_1 ; \lambda, \mu) = \sum_{\nu} N^{\nu}_{\lambda\mu} J(K ; \nu)$ where $N^{\nu}_{\lambda\mu}$ are the structure constants of the fusion algebra $R_K$.

Define generalized notion of J-polynomial by considering invariant $J(L; x_1, \dots, x_m)$ with $x_1, \dots, x_m \in R_K$. For $x_j = v_{\lambda_j}$ for $j = 1, \dots, m$,

$J(L; x_1, \dots, x_m) = J_L(L; \lambda_1, \dots, \lambda_m)$

Then for $x_j = v_\lambda \cdot v_\mu$ take

$$J(L; \dots, v_\lambda \cdot v_\mu, \dots) = \sum_\nu N_{\lambda\mu}^\nu \, J(K; \dots v_\nu \dots).$$

$\longrightarrow$ obtain multi-linear map

$$J(L) : R_K^{\otimes m} \longrightarrow \mathbb{C}$$

Proposition 3:

For links $L_1$ and $L_2$ contained in disjoint 3-balls $B_1$ and $B_2$ respectively, we have

$$J(L_1 \cup L_2; \mu_1, \mu_2) = J(L_1; \mu_1) \, J(L_2; \mu_2)$$

Proof:

In the construction of $Z(L_1 \cup L_2; \mu_1 \cup \mu_2)$ put $B_1$ and $B_2$ in such a way that

$$Z(L_1 \cup L_2; \mu_1 \cup \mu_2) = Z(L_1; \mu_1) \circ Z(L_2; \mu_2)$$
$$= Z(L_1; \mu_1) \, Z(L_2; \mu_2)$$

$\longrightarrow$ correct by factors of $d(\mu_i)$ to obtain result $\square$

**Definition:**

We denote by $\bar{L}$ the mirror image of $L$.
("look from the other side of the blackboard")

**Proposition 4:**

Let $L$ be an oriented framed link. For the mirror image $\bar{L}$ we have

$$J(\bar{L};\lambda) = \overline{J(L;\lambda)}$$

where the right hand side stands for the complex conjugate of $J(L;\lambda)$.

**Proof:**

The monodromy matrix $\rho(\sigma^{-1})$ is obtained from $\rho(\sigma)$ by replacing $q$ with $q^{-1}$. The entries of connection matrix $F$ and $d(\lambda)$ are real $\longrightarrow$ $J_{\bar{L}}(q) = J_L(q^{-1})$.

since $q$ is root of unity

$$\longrightarrow J(\bar{L},\lambda) = \overline{J(L;\lambda)}$$

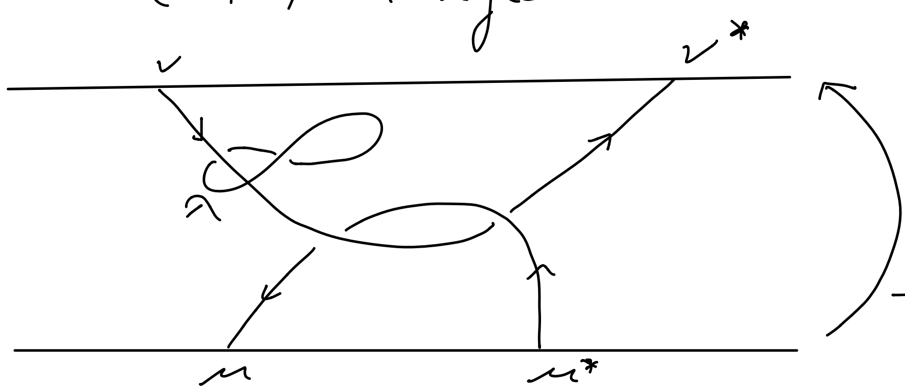
$\square$

## Oriented framed tangles:

Set $X = \mathbb{C} \times [0,1]$

Let $p_1, \dots, p_m$ be $m$ distinct points on the real line of $X_0 = \mathbb{C} \times \{0\}$ and let $q_1, \dots, q_n$ be $n$ distinct points on real line of $X_1 = \mathbb{C} \times \{0\}$.

A compact 1-manifold $T$ in $X$ with boundary $\{p_1, \dots, p_m, q_1, \dots, q_n\}$ is called an $(m,n)$-"tangle"




$$Z(T;\lambda): V_{\mu_1 \dots \mu_m} \longrightarrow V_{\nu_1 \dots \nu_n}$$

Similarly, we get a linear map

$$J(T;\lambda): V_{\mu_1 \dots \mu_m} \longrightarrow V_{\nu_1 \dots \nu_n}$$

"tangle operator"